

On $(k \oplus l|q)$ -dimensional supermanifolds.

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Abstract

We define a $(k \oplus l|q)$ -dimensional supermanifold as a manifold having q odd coordinates and $k + l$ even coordinates with l of them taking only nilpotent values. We show that this notion can be used to formulate superconformal field theories with different number of supersymmetries in holomorphic and antiholomorphic sectors.

The present preprint is an extended version of paper [1]. It stems from an attempt to understand the precise mathematical meaning of some constructions used by physicists. We have in mind in particular the consideration of independent spin structures in holomorphic and antiholomorphic sectors of N=1 superconformal field theory and the notion of chiral (heterotic) supermanifold (for example see [2]). In this paper we introduce a notion of $(k \oplus l|q)$ -dimensional supermanifold and show that in many cases rigorous definitions can be based on this notion. Mathematical details are relegated to the appendix that will be published separately. The appendix also contains the details of formulation of supergeometry in terms of functors. This language is most suitable for our purposes and is convenient in many other supergeometry questions as well.

Let $\Lambda = \Lambda_0 + \Lambda_1$ be a Grassmann algebra with an even subspace Λ_0 and an odd subspace Λ_1 . The space $\mathbf{R}_\Lambda^{p|q}$ can be defined as a space consisting of rows $(x^1, \dots, x^p, \xi^1, \dots, \xi^q)$ where $x^1, \dots, x^p \in \Lambda_0$ are even elements of Grassmann algebra Λ and $\xi^1, \dots, \xi^q \in \Lambda_1$ are odd elements from Λ . Physicists usually say that one can take as Λ any Grassmann algebra provided it is large enough. From the viewpoint of a mathematician it is better to consider a family of sets $\mathbf{R}_\Lambda^{p|q}$ corresponding to all Grassmann algebras Λ . It is easy to see that a parity preserving homomorphism $\alpha : \Lambda \rightarrow \Lambda'$ generates naturally a map $\tilde{\alpha} : \mathbf{R}_\Lambda^{p|q} \rightarrow \mathbf{R}_{\Lambda'}^{p|q}$ and that $\widetilde{\beta\alpha} = \tilde{\beta}\tilde{\alpha}$ for any parity preserving homomorphisms

$\alpha : \Lambda \rightarrow \Lambda', \beta : \Lambda' \rightarrow \Lambda''$. In the language of mathematics this means that the correspondence $\Lambda \mapsto \mathbf{R}_\Lambda^{p|q}$ determines a functor acting from the category of Grassmann algebras into the category of sets (or of vector spaces).

An (even) superfield on $\mathbf{R}^{p|q}$ can be defined as an expression of the form

$$\sum_{k=2l} \sum_{1 \leq i_1 < \dots < i_k} f_{i_1, \dots, i_k}(x^1, \dots, x^p) \xi^{i_1} \dots \xi^{i_k} \quad (1)$$

Here f_{i_1, \dots, i_k} are smooth functions on \mathbf{R}^p . It is important to notice that such an expression determines a map $F_\Lambda : \mathbf{R}_\Lambda^{p|q} \rightarrow \Lambda_0 = \mathbf{R}_\Lambda^{1|0}$. This follows from the fact that we can substitute an even element of a Grassmann algebra into any smooth function of real variable. To verify this statement we notice that every even element x of Grassmann algebra can be represented in the form $x = m + n$ where $m \in \mathbf{R}$ and n is nilpotent. We define $f(x)$ using the Taylor expansion $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(m)}{k!} n^k$ (the series terminates because n is nilpotent). If $\alpha : \Lambda \rightarrow \Lambda'$ is a parity preserving homomorphism then $\tilde{\alpha} \circ F_\Lambda = F_{\Lambda'} \circ \tilde{\alpha}$. This means in mathematical terminology that a superfield (1) specifies a natural map of the functor $\mathbf{R}^{p|q}$ into the functor $\mathbf{R}^{1|0}$. Analogously, an odd superfield can be considered as a natural transformation of functors $\mathbf{R}^{p|q} \rightarrow \mathbf{R}^{0|1}$. An arbitrary superfield can be viewed as a natural transformation of functors $\mathbf{R}^{p|q} \rightarrow \mathbf{R}^{1|1}$ (recall that $\mathbf{R}_\Lambda^{1|1} = \Lambda_0 \oplus \Lambda_1$). Note that in the above considerations the Grassmann algebra can be replaced by any algebra Λ every element of which can be represented as a sum of a real number and nilpotent element (assuming that Λ is associative, \mathbb{Z}_2 -graded, supercommutative algebra having unit element). Algebras of this kind will be called almost nilpotent algebras or AN algebras. In other words we can say that $\mathbf{R}^{p|q}$ can be considered as a functor on the category of AN algebras and a superfield of definite parity determines a natural transformation of the functor $\mathbf{R}^{p|q}$ into the functor $\mathbf{R}^{1|0}$ or $\mathbf{R}^{0|1}$.

The main notions of superalgebra and of supergeometry can be formulated very easily in the language of functors. A superspace can be defined as an arbitrary functor on the category **AN** of AN algebras taking values in the category of sets. The body of a superspace can be defined as the set corresponding to the AN algebra $\Lambda = \mathbf{R}$. We introduce a $(p|q)$ -dimensional supermanifold as a superspace that is locally equivalent to $\mathbf{R}^{p|q}$. In other words $(p|q)$ -dimensional supermanifold can be pasted together from domains in $\mathbf{R}^{p|q}$ by means of smooth transformations. A body of a $(p|q)$ -dimensional supermanifold is a p -dimensional smooth manifold. Replacing in the definition of a superspace the category of sets by the category of groups or by the category of Lie algebras we obtain the definitions of supergroup and super Lie algebra respectively.

It is convenient to generalize the notion of the superspace $\mathbf{R}^{p|q}$ as follows. Denote by $\mathbf{R}_\Lambda^{k \oplus l|q}$ a set of rows $(x^1, \dots, x^k, y^1, \dots, y^l, \xi^1, \dots, \xi^q)$ where x^1, \dots, x^k are arbitrary even elements of AN algebra Λ , y^1, \dots, y^l are nilpotent even elements of Λ and ξ^1, \dots, ξ^q are odd elements of Λ . A superspace $\mathbf{R}^{k \oplus l|q}$ can be defined as a functor $\Lambda \rightarrow \mathbf{R}_\Lambda^{k \oplus l|q}$. Strictly speaking to define a functor we

should also construct a homomorphism $\tilde{\alpha} : \mathbf{R}_\Lambda^{k \oplus l|q} \rightarrow \mathbf{R}_{\Lambda'}^{k \oplus l|q}$ for every parity preserving homomorphism $\alpha : \Lambda \rightarrow \Lambda'$ of AN algebras. We omit this obvious construction.

Let us define a superfield on $\mathbf{R}^{k+l|q}$ as an expression of the form

$$\sum_s \sum_{1 \leq i_1 < \dots < i_s} f_{i_1, \dots, i_s}(x^1, \dots, x^k, y^1, \dots, y^l) \xi^{i_1} \dots \xi^{i_s} \quad (2)$$

where f_{i_1, \dots, i_s} are smooth functions of variables $(x^1, \dots, x^k) \in \mathbf{R}^k$ and formal power series with respect to y^1, \dots, y^l . Such an expression determines a map of $\mathbf{R}^{k \oplus l|q}$ into $\mathbf{R}^{1|0}$ if expression (2) is even and into $\mathbf{R}^{0|1}$ if it is odd.

We define a $(k \oplus l|q)$ -dimensional supermanifold as a superspace that is locally equivalent to $\mathbf{R}^{k \oplus l|q}$. Almost all notions of (super)geometry can be generalized to the case of $(k \oplus l|q)$ -dimensional supermanifolds. In particular one can define the supergroup of transformations of such a manifold and corresponding Lie (super) algebra of vector fields. As usual, a vector field on a $(k \oplus l|q)$ -dimensional superdomain with coordinates $(x^1, \dots, x^k, y^1, \dots, y^l, \xi^1, \dots, \xi^q)$ can be identified with a first order differential operator

$$A^i \frac{\partial}{\partial x^i} + B^j \frac{\partial}{\partial y^j} + C^s \frac{\partial}{\partial \xi^s} \quad (3)$$

where A^i, B^j, C^s are smooth functions of variables x^1, \dots, x^k , formal power series with respect to y^1, \dots, y^l and polynomials in ξ^1, \dots, ξ^q . If A^i, B^j are odd and C^s are even then the operator (3) is parity reversing; we say that the corresponding vector field is odd. The definition of an even vector field is similar. A $(k \oplus l|q)$ -dimensional supermanifold is pasted together from $(k \oplus l|q)$ -dimensional superdomains by means of smooth transformations. If we have a $(p|q)$ -dimensional supermanifold \mathcal{A} and a k -dimensional submanifold $\mathcal{B}^{(0)}$ of its body $\mathcal{A}^{(0)}$ we can construct easily a $(k \oplus (p-k)|q)$ -dimensional supermanifold in the following way. For each Λ there is a natural mapping $\tilde{m}_\Lambda : \mathcal{A}_\Lambda \rightarrow \mathcal{A}^{(0)}$ corresponding to the homomorphism $m_\Lambda : \Lambda \rightarrow \mathbf{R}$ that evaluates a numerical part of an element from Λ . Then, \mathcal{B}_Λ consists of those elements of \mathcal{A}_Λ that project onto $\mathcal{B}^{(0)}$ under \tilde{m}_Λ . The mapping $\tilde{\alpha} : \mathcal{B}_\Lambda \rightarrow \mathcal{B}_{\Lambda'}$ assigned to a homomorphism $\alpha : \Lambda \rightarrow \Lambda'$ is a restriction of the corresponding map defined for the supermanifold \mathcal{A} . Let us call the supermanifold \mathcal{B} obtained this way the restriction of supermanifold \mathcal{A} to the subset \mathcal{B}_0 of its body.

One can define classes of supermanifolds with interesting geometric properties restricting the allowed class of coordinate transformations. For example one can define a complex analytic transformation of a $(2k \oplus 2l|2q)$ -dimensional superdomain generalizing the usual requirement that the Jacobian matrix of transformation commutes with a standard matrix J obeying $J^2 = -1$. Then, a $(k \oplus l|q)$ -dimensional complex manifold can be defined as a manifold glued together by complex analytic transformations. One can also use complex coordinates $(Z^A) = (z^1, \dots, z^k, w^1, \dots, w^l, \vartheta^1, \dots, \vartheta^q)$ on a complex manifold. They

take values in complex AN algebras with antilinear involution. Of course together with these coordinates we should consider complex conjugate coordinates \bar{Z}^A . Analytic transformations do not mix Z^A and \bar{Z}^A . Therefore usually we will not mention \bar{Z}^A .

Let U be a $(1|N)$ -dimensional complex superdomain with complex coordinates $(z, \theta^1, \dots, \theta^N)$. We define N-superconformal transformations as complex analytic transformations preserving up to a factor the one-form $\omega_N = dz + \theta^1 d\theta^1 + \dots + \theta^N d\theta^N$ (up to a factor means here up to multiplication by a nonvanishing superfield). For example in case $N = 1$ superconformal transformations can be written explicitly as follows

$$\begin{aligned}\tilde{z} &= u(z) - u'(z)\epsilon(z)\theta \\ \tilde{\theta} &= \sqrt{u'(z)} \left(\theta + \epsilon(z) + \frac{1}{2}\epsilon(z)\epsilon'(z)\theta \right)\end{aligned}\tag{4}$$

where $u(z)$ and $\epsilon(z)$ are even and odd analytic functions of z respectively. By a N-superconformal manifold we mean a manifold pasted together from $(1|N)$ -dimensional complex superdomains by N-superconformal transformations. Note that strictly speaking we are looking here not at a single supermanifold but rather at a family of supermanifolds parameterized by gluing functions similar to $u(z)$ and $\epsilon(z)$ in $N = 1$ case. The superspace M_N of equivalence classes of N-superconformal manifolds is called a moduli space of N-superconformal structures (supermoduli space).

An N-superconformal vector field defined in a complex superdomain $U^{1|N}$ is a vector field X such that the Lie derivative of the form ω_N restricted to $U^{1|N}$ with respect to X is proportional to ω_N , i.e. $L_X \omega_N = f \omega_N$ for some superfield f . Such vector fields correspond to infinitesimal N-superconformal transformations. Given an N-superconformal compact manifold \mathcal{A} one can define an N-superconformal vector field on it as a vector field such that its restriction to each elementary coordinate patch $U^{1|N}$ is N-superconformal. It follows from standard results of deformation theory that a formal tangent space to the moduli space of N-superconformal structures at a “point” \mathcal{A} (being an N-superconformal manifold) is isomorphic to $H^1(\mathcal{A}, \gamma_N)$. Here $H^1(\mathcal{A}, \gamma_N)$ stands for the first cohomology group of \mathcal{A} with coefficients in the sheaf of N-superconformal vector fields γ_N .

Moduli spaces play the central role in the Segal’s axiomatics of conformal field theory (CFT) ([6]). In this approach one considers conformal 2d surfaces (complex curves) with parametrized boundary components. Each boundary component is homeomorphic to a circle and it is assumed that the parametrization can be extended to a complex coordinate in a small neighborhood. We can think of a standard annulus in the complex plane $\{z \in \mathbf{C} | \frac{1}{2} \leq |z| \leq 1\}$ mapped by a biholomorphic mapping into a neighborhood of the boundary component. The neighborhoods of different boundary components are assumed to be nonoverlapping. The annuli are divided into two classes: “incoming” and

“outgoing”. The moduli space of such objects with m incoming and n outgoing annuli is denoted by $P_{m,n}$. Let us stress here that we allow disconnected surfaces as well. There arise naturally two operations on the sets $P_{m,n}$:

$$P_{m_1,n_1} \times P_{m_2,n_2} \rightarrow P_{m_1+n_1,m_2+n_2} \quad (5)$$

$$P_{m,n} \rightarrow P_{m-1,n-1} \quad (6)$$

Here the first operation corresponds to the disjoint union of surfaces and the second one corresponds to the identification of m -th incoming annulus with the n -th outgoing one by the rule: $z' = \frac{1}{4}z^{-1}$. In Segal’s axiomatics CFT is specified by maps

$$\alpha_{m,n} : P_{m,n} \rightarrow \text{Hom}_{\mathbf{C}}\{H^m, H^n\}$$

assigning to each point in $P_{m,n}$ a linear mapping $H^m \rightarrow H^n$ belonging to the trace class (here H is a fixed Hilbert space). The collection of mappings $\alpha_{m,n}$ should satisfy some set of axioms ensuring the compatibility of mappings $\alpha_{m,n}$ with mappings (5), (6) and with permutations of boundary components. One can generalize Segal’s axiomatics to the case of N-superconformal field theories replacing 2d conformal surfaces by N-superconformal manifolds with boundaries and complex annuli by N-superconformal annuli (see [5] for this type of axiomatics stated for N=2 SCFT).

Now let us describe supermanifolds that appear in 2D superconformal field theories (SCFT) having different number of supersymmetries for left movers and right movers. Consider a $(2|p+q)$ -dimensional complex domain U with even coordinates z_L, z_R and odd coordinates $\theta_L^1, \dots, \theta_L^p, \theta_R^1, \dots, \theta_R^q$. The superdomain U can be considered as a superdomain in real superspace $\mathbf{R}^{4|2p+2q}$. We will single out a subspace V of U that has real dimension $(2 \oplus 2|2p+2q)$ by imposing the condition that $z_R - \bar{z}_L$ is nilpotent. By definition a transformation of V is called (p, q) -superconformal if it does not mix the left coordinates z_L, θ_L^i with the right coordinates z_R, θ_R^i and preserves up to a factor one-forms

$$\begin{aligned} \omega_L &= dz_L + \theta_L^1 d\theta_L^1 + \dots + \theta_L^p d\theta_L^p \\ \omega_R &= dz_R + \theta_R^1 d\theta_R^1 + \dots + \theta_R^q d\theta_R^q \end{aligned}$$

We define a (p, q) -superconformal manifold as a superspace pasted together from several copies of V by means of (p, q) -superconformal transformations. Again we assume here that the odd gluing parameters such as $\epsilon(z)$ in (4) are allowed.

Let $\mathcal{A}^{(0)}$ be a compact connected oriented 2-dimensional surface of genus $g > 1$. It is easy to define a moduli space $\mathcal{M}_{p,q,g}$ of (p, q) -superconformal manifolds having $\mathcal{A}^{(0)}$ as a body. Denote by $\mathcal{M}_{p,q,g}^{(0)}$ the body of moduli space $\mathcal{M}_{p,q,g}$. One can prove that the superspace $\mathcal{M}_{p,q,g}$ can be constructed out of supermoduli spaces $\mathcal{M}_{p,g}$ and $\mathcal{M}_{q,g}$ of p and q -superconformal manifolds with body $\mathcal{A}^{(0)}$. The construction goes as follows. As in the case of moduli space of conformal structures $\mathcal{M}_{p,g}$ is equipped with a canonical complex structure. Thus we may consider the superspace $\mathcal{M}_{p,g} \times \bar{\mathcal{M}}_{q,g}$ where $\bar{\mathcal{M}}_{q,g}$ denotes the

space $\mathcal{M}_{q,g}$ with conjugate complex structure. Note that there is a natural projection $\pi_p : \mathcal{M}_{p,g} \rightarrow \mathcal{M}_g$ to the moduli space of complex structures on $\mathcal{A}^{(0)}$. This projection simply corresponds to the fact that each p -superconformal manifold has a complex structure on its body. If z is a complex coordinate on \mathcal{M}_g and \bar{z} is its counterpart on $\bar{\mathcal{M}}_g$ then we define the subset $\mathcal{M}_{p,q,g}^{(0)} \subset \mathcal{M}_{p,g}^{(0)} \times \bar{\mathcal{M}}_{q,g}^{(0)}$ as the set of points (a, b) such that $\pi_p(a) = (\pi_q(b))^*$. Here $*$ stands for the conjugated complex structure. The superspace $\mathcal{M}_{p,q,g}$ has the body $\mathcal{M}_{p,q,g}^{(0)}$ and can be obtained as a restriction of $\mathcal{M}_{p,g} \times \bar{\mathcal{M}}_{q,g}$ to the corresponding subset of its body. Choosing a local coordinate system on $\mathcal{M}_{p,g} \times \bar{\mathcal{M}}_{q,g}$ in such a way that z, \bar{z} are part of the coordinates we see that the condition above implies that $z - (\bar{z})^*$ can take only nilpotent values (as this is zero on the body). This means that if $\mathcal{M}_{p,g}$ is of dimension $(k|l)$ and $\mathcal{M}_{q,g}$ is of dimension $(\tilde{k}|\tilde{l})$ the superspace $\mathcal{M}_{p,q,g}$ has the dimension $((k + \tilde{k} - 3g + 3) \oplus (3g - 3)|l + \tilde{l})$. The construction of $\mathcal{M}_{p,q,g}$ presented above is simply a more formal way to say that left and right superconformal structures on a (p, q) -superconformal manifold are independent up to a complex structure on the body that they share. It is easy to generalize this construction to the case when \mathcal{A}_0 is disconnected.

Now we are in a position to generalize Segal's axiomatics to include (p, q) -superconformal field theories. To define a proper analog of the space $P_{m,n}$ one should consider a larger moduli space of (not necessarily connected) (p, q) -superconformal surfaces having parameterized boundary components of different type (determined by the boundary conditions for odd coordinates). Furthermore, the corresponding analogs of the mappings $\alpha_{m,n}$ should be holomorphic. The last requirement sets a connection between p -superconformal and (p, p) -superconformal theories as outlined below. Firstly, it is possible to embed the supermoduli space $\mathcal{M}_{p,g}$ into $\mathcal{M}_{p,p,g}$. Then, given a real analytic function on $\mathcal{M}_{p,g} \subset \mathcal{M}_{p,p,g}$ one can extend it to a holomorphic function defined in some neighborhood of $\mathcal{M}_{p,g}$ in $\mathcal{M}_{p,p,g}$. Assuming that a continuation to the whole $\mathcal{M}_{p,p,g}$ is possible we see that a p -superconformal field theory determines a (p, p) -superconformal one.

Now let us make some remarks about the integration over $\mathcal{M}_{p,q,g}$. This question is important in particular in heterotic string theory. To calculate a string amplitude one defines a holomorphic volume element on $\mathcal{M}_{p,q,g} \times \bar{\mathcal{M}}_{p,q,g}$ and chooses a real cycle of integration on the body of this space. We would like to stress that possible reality conditions for odd and nilpotent variables do not affect the integration result (see [7] or [8] for discussion). Simply an integration over odd variables is essentially an algebraic operation. As for the choice of real cycle on the body of $\mathcal{M}_{p,q,g} \times \bar{\mathcal{M}}_{p,q,g}$ one can take the (real) diagonal of the body of $\mathcal{M}_{p,q,g} \times \bar{\mathcal{M}}_{p,q,g}$. The change of nilpotent variable roughly speaking corresponds to the infinitesimal change of integration cycle and therefore does not change the value of integral.

Appendix

In this appendix we develop the approach to definition of superspace in terms of functors (see [3]).

The notion of superspace. Let us introduce a notion of almost nilpotent algebra (or AN algebra). An AN algebra is an associative finite dimensional Z_2 -graded supercommutative algebra A with unit element such that its ideal of nilpotent elements has codimension 1. In other words A can be decomposed (as a vector space) in a direct sum $A = R + N$ where R is the canonically embedded ground field (\mathbf{R} or \mathbf{C}) and $N = N(A)$ consists of nilpotent elements. An example of AN algebra is a Grassmann algebra. Another important example is an algebra generated by unit element 1 and an element x satisfying the only relation $x^n = 0$. An example including the previous two is a supercommutative algebra with odd generators $\xi_1, \xi_2, \dots, \xi_n$ and even generators x_1, x_2, \dots, x_m satisfying the relations $x_1^{n_1} = 0, x_2^{n_2} = 0, \dots, x_m^{n_m} = 0$. Below parity preserving homomorphisms of AN algebras are called morphisms.

Now we define a superspace S as a rule assigning to each AN algebra Λ a set S_Λ that we call the set of Λ -points of S and to each morphism of two AN algebras $\alpha : \Lambda \rightarrow \Lambda'$ a map $\tilde{\alpha} : S_\Lambda \rightarrow S_{\Lambda'}$ of the corresponding sets in a way that is consistent with compositions of morphisms. The last assertion means that the map $\widetilde{\alpha_2 \alpha_1}$ corresponding to the composition of morphisms $\alpha_2 \alpha_1$ is equal to the composition of maps $\tilde{\alpha}_2 \tilde{\alpha}_1$. We consider all AN algebras together with parity preserving homomorphisms between them as a category **AN**. In standard mathematical terminology superspace is a covariant functor from the category **AN** to the category of sets. The space S_R corresponding to an AN algebra R (i.e. an AN algebra having a trivial nilpotent part N) is called the body of superspace S . For every AN algebra Λ there is a homomorphism $m : \Lambda \rightarrow R$ assigning to an element of Λ its projection onto R with respect to the canonical decomposition $\Lambda = R + N(\Lambda)$. The corresponding map $\tilde{m} : S_\Lambda \rightarrow S_R$ associates with each Λ -point $a \in S_\Lambda$ a point $\tilde{m}(a)$ in the body of S that we call the numeric part of a . From now on if not specified we will assume our ground field is the field of real numbers \mathbf{R} . Here we give some basic examples of superspaces.

Example 1 Let V be a Z_2 graded vector space, i.e. $V = V_0 \oplus V_1$ where V_0 and V_1 are called the even and odd subspaces respectively. We define the set V_Λ to be the set of all formal (finite) linear combinations $\sum_i e_i \alpha_i + \sum_j f_j \beta_j$ where $e_i \in V_0, f_j \in V_1$, α_i and β_j are respectively arbitrary even and odd elements from Λ (we assume the following identifications $(\alpha_1 + \alpha_2)v = \alpha_1 v + \alpha_2 v$, $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$ where $\alpha, \alpha_1, \alpha_2 \in \Lambda$ and $v, v_1, v_2 \in V$). The mapping $\tilde{\rho}$ corresponding to a morphism $\rho : \Lambda \rightarrow \Lambda'$ carries the point $\sum_i e_i \alpha_i + \sum_j f_j \beta_j$ to $\sum_i e_i \rho(\alpha_i) + \sum_j f_j \rho(\beta_j)$ that obviously lies in $V_{\Lambda'}$. One can easily see that the maps $\tilde{\rho}$ behave properly under compositions. Note that each set V_Λ is a linear space itself and is equipped with a canonical structure of (left or right) Λ_0 module where Λ_0 is an even part of Λ . For the last statement we set $\mu(\sum_i e_i \alpha_i + \sum_j f_j \beta_j) = \sum_i e_i (\mu \alpha_i) + \sum_j f_j (\mu \beta_j)$ for any $\mu \in \Lambda_0$ (this is for

the left Λ_0 module structure, the modification for the structure of right Λ_0 module is obvious). We call superspaces with this property linear superspaces. The construction given above works for any Z_2 -graded linear space, including infinite-dimensional ones. In the finite-dimensional case the corresponding superspace is called $(p|q)$ -dimensional linear superspace and is denoted by $\mathbf{R}^{p|q}$ (here $p = \dim V_0, q = \dim V_1$). Choosing a basis in V_0 and V_1 one obtains a representation of $\mathbf{R}_\Lambda^{p|q}$ as a set of $(p+q)$ -tuples of elements from Λ where first p elements are from Λ_0 and the next q elements are from Λ_1 . It readily follows that the body of $\mathbf{R}^{p|q}$ is naturally isomorphic (as a linear space) to \mathbf{R}^p .

Example 2 Here we want to describe some general operation over superspaces. Let \mathcal{M} be a superspace with a body $\mathcal{M}^{(0)} \equiv \mathcal{M}_R$. Let $\mathcal{N}^{(0)} \subset \mathcal{M}^{(0)}$ be any subset. Define $\mathcal{N}_\Lambda = (\tilde{m}(\mathcal{N}^{(0)}))^{-1} \subset \mathcal{M}_\Lambda$ where \tilde{m} is the described above map evaluating the numerical part of Λ -point. For any morphism of AN algebras $\rho : \Lambda \rightarrow \Lambda'$ we can define a map $\mathcal{N}_\Lambda \rightarrow \mathcal{N}_{\Lambda'}$ as the restriction of $\tilde{\rho}$ to \mathcal{N}_Λ . Then it is easy to check that the sets \mathcal{N}_Λ together with maps $\tilde{\rho}$ form a superspace. This new superspace has the set $\mathcal{N}^{(0)}$ as its body. The superspace constructed this way out of the given superspace \mathcal{M} and a subset $\mathcal{N}^{(0)} \subset \mathcal{M}^{(0)}$ will be called the restriction of superspace \mathcal{M} to the set $\mathcal{N}^{(0)}$. Given a domain $U \subset \mathbf{R}^p$ we can construct a restriction of superspace $\mathbf{R}^{p|q}$ to U which is called a $(p|q)$ -dimensional superdomain and denoted as $U^{p|q}$.

Example 3 Given a linear subspace $\mathbf{R}^m \subset \mathbf{R}^p$ one can construct the restriction of superspace $\mathbf{R}^{p|q}$ to this subset. We denote this new superspace by $\mathbf{R}^{m \oplus n | q}$ where $n = p - m$. The explicit construction of it is as follows. Let $V = V_0 \oplus V_1$ be a Z_2 graded vector space and $\dim V_0 = p, \dim V_1 = s$. Given a decomposition $V_0 = U_0 \oplus W_0$ ($\dim U_0 = m, \dim W_0 = n$) we define $\mathbf{R}_\Lambda^{m \oplus n | s}$ to be the set of linear combinations of the form $\sum_i d_i \alpha_i + \sum_j e_j \beta_j + \sum_k f_k \gamma_k$ where $d_i \in U_0, e_j \in W_0, f_k \in V_1$ and $\alpha_i \in \Lambda_0, \beta_j \in \Lambda_0 \cap N(\Lambda), \gamma_k \in \Lambda_1$ (as above we require the natural distributivity conditions). Given bases of U_0, W_0, V_1 we can represent a point from $\mathbf{R}_\Lambda^{m \oplus n | s}$ as a $p+q$ -tuple of elements from Λ in which first m elements are even, next n elements are even and nilpotent and the last s elements are odd. Similarly to the construction of $\mathbf{R}^{p|q}$ for any morphism of AN algebras ρ we can construct a mapping $\tilde{\rho}$ that carries the point $\sum_i d_i \alpha_i + \sum_j e_j \beta_j + \sum_k f_k \gamma_k$ to $\sum_i d_i \rho(\alpha_i) + \sum_j e_j \rho(\beta_j) + \sum_k f_k \rho(\gamma_k)$. Given a domain $U \subset \mathbf{R}^m$ the restriction of $\mathbf{R}^{m \oplus n | s}$ to U is called an $(m \oplus n | s)$ -dimensional superdomain and is denoted by $U^{m \oplus n | s}$. Note that like $\mathbf{R}^{p|q}$, $\mathbf{R}^{m \oplus n | q}$ is a vector superspace.

In a more interesting situation we have a superspace \mathcal{M} such that the sets \mathcal{M}_Λ are equipped with some additional structure and the maps $\tilde{\rho}$ are consistent with it. In other words we can consider a functor from the category **AN** to a category of sets with additional structure, e.g. smooth manifolds, groups, Lie groups, Lie algebras, etc. In the previous section we gave an example of this sort of object, namely a vector superspace is a functor with values in linear spaces. Similarly, one can define a topological superspace as a functor from the category

AN to the category of topological spaces. We define a smooth superspace \mathcal{M} as a functor with values in smooth manifolds satisfying some additional conditions to be described in a moment. Recall that in the examples described above spaces of Λ -points of linear superspaces were equipped with a structure of Λ_0 -module. Thus we will require that a tangent space at each point in \mathcal{M}_Λ is equipped with a structure of (left) Λ_0 -module. A mapping $\tilde{\alpha} : M_\Lambda \rightarrow M_{\Lambda'}$ corresponding to a morphism α of two AN algebras is called Λ_0 -smooth if the corresponding tangent map $(\tilde{\alpha})_*$ commutes with Λ_0 -module structures, i.e. $(\tilde{\alpha})_* \circ \lambda_0 = \alpha(\lambda_0) \circ (\tilde{\alpha})_*$.

Next we introduce the notions of associative superalgebra, Lie superalgebra and supergroup. An associative superalgebra is a functor \mathcal{A} taking values in the category of associative algebras. In addition we require that each algebra $\mathcal{A}(\Lambda)$ is equipped with a structure of Λ_0 -module in a way that is consistent with associative multiplication and the mappings $\tilde{\alpha}$. An example of an associative superalgebra can be obtained in the following way. Given a Z_2 -graded associative algebra A one can consider linear combinations whose coefficients are elements of AN algebra Λ with appropriate parity (as in the construction of vector superspace). Setting $(a\lambda)(b\mu) = \pm(ab)(\lambda\mu)$ for all $(a, b) \in A$, $\lambda, \mu \in \Lambda$ (where minus sign occurs only if λ and b are both odd) we get an associative algebra associated with every A and AN algebra Λ . These associative algebras can be considered as algebras on sets of Λ -points of a certain superspace. A Lie superalgebra is a functor with values in Lie algebras equipped with a natural structure of linear superspace. Analogously to the previous example one can construct a Lie superalgebra out of any Z_2 -graded Lie algebra. Another example is a Lie superalgebra constructed out of an associative superalgebra. One simply defines a Lie algebra structure on the sets of Λ -points via the commutator.

A supergroup is a functor with values in the category of groups. An important example is a supergroup $GL(m|n)$ whose set of Λ -points consists of $(m+n) \times (m+n)$ nondegenerate block matrices with entries from Λ . The entries of $m \times m$ and $n \times n$ blocks are even and the entries of $m \times n$ and $n \times m$ ones are odd. A matrix is called nondegenerate if the left inverse matrix exists (one can show that this implies the existence of right inverse matrix). The set of all such block matrices (not necessarily nondegenerate) is an example of associative superalgebra.

We say that a supergroup G acts on a superspace \mathcal{S} if for any AN algebra Λ the group G_Λ acts on the set \mathcal{S}_Λ in a way that is consistent with mappings $\tilde{\alpha}$ between Λ -points. Namely, if ϕ_g is the mapping of the set \mathcal{S}_Λ to itself corresponding to the element $g \in G_\Lambda$, then $\tilde{\alpha} \circ \phi_g = \phi_{\tilde{\alpha}(g)} \circ \tilde{\alpha}$. The collection of quotient spaces $\mathcal{S}_\Lambda/G_\Lambda$ is endowed with the structure of superspace in a natural way.

Supermanifolds. In this section we would like to give a definition of a supermanifold as a superspace with values in the category of smooth manifolds which is glued together from $(m \oplus n|s)$ -dimensional superdomains. To this end we need to describe an appropriate class of gluing transformations.

We start with the definition of a mapping of one superspace into another. Let \mathcal{N} and \mathcal{M} be superspaces. We say that there is a mapping $F : \mathcal{N} \rightarrow \mathcal{M}$ if for every AN algebra Λ there is a map $F_\Lambda : \mathcal{N}_\Lambda \rightarrow \mathcal{M}_\Lambda$ and the maps F_Λ are consistent with maps $\tilde{\alpha}$ corresponding to the morphisms of AN algebras $\alpha : \Lambda \rightarrow \Lambda'$. In other words diagrams of the following type are commutative

$$\begin{array}{ccc} \mathcal{N}_\Lambda & \xrightarrow{F_\Lambda} & \mathcal{M}_\Lambda \\ \tilde{\alpha} \downarrow & & \downarrow \tilde{\alpha} \\ \mathcal{N}_{\Lambda'} & \xrightarrow{F_{\Lambda'}} & \mathcal{M}_{\Lambda'} \end{array} \quad (7)$$

In standard terminology a mapping of superspaces is a natural transformation of functors. Our next goal will be to describe all possible mappings $F : \mathbf{R}^{p+q|s} \rightarrow \mathbf{R}^{p'+q'|s'}$. Note that for now we do not require our maps to be consistent with natural Λ_0 -module structure defined on $\mathbf{R}_\Lambda^{p+q|s}$ and $\mathbf{R}_\Lambda^{p'+q'|s'}$, nor do we impose any smoothness conditions on F_Λ . We introduce an algebra $A_{p,q,s}$ consisting of the expressions of the form

$$\omega = \sum_{i=0}^s \sum_{1 \leq \alpha_1 < \dots < \alpha_i \leq s} f^{\alpha_1, \dots, \alpha_i} \xi_{\alpha_1} \dots \xi_{\alpha_i} \quad (8)$$

where

$$f^{\alpha_1, \dots, \alpha_i} = \sum_{(\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_q)} G_{\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_q}^{\alpha_1, \dots, \alpha_i} (m_1, \dots, m_p) n_1^{\beta_1} \dots n_p^{\beta_p} l_1^{\gamma_1} \dots l_q^{\gamma_q} \quad (9)$$

are formal series in variables $n_1, \dots, n_p, l_1, \dots, l_q$, the coefficients of which are arbitrary functions of real variables m_1, \dots, m_p . If one assumes that ξ_1, \dots, ξ_s in (8) are generators of Grassmann algebra one can define the parity of expression (8) in a natural way. Given an even (odd) element $\omega \in A_{p,q,s}$ one can construct a mapping α_Λ^ω of $\mathbf{R}_\Lambda^{p+q|s}$ into $\mathbf{R}_\Lambda^{1|0}$ (respectively, into $\mathbf{R}_\Lambda^{0|1}$) in the following way. Given a Λ -point $X = (x_1, \dots, x_p, y_1, \dots, y_q, \theta_1, \dots, \theta_s) \in \mathbf{R}_\Lambda^{p+q|s}$ to evaluate $\alpha_\Lambda^\omega(X)$ one should substitute in (8), (9) odd elements θ_i for ξ_i , numeric part $m(x_i)$ for m_i , nilpotent part $x_i - m(x_i)$ for n_i and y_i for l_i . Since y_i and $x_i - m(x_i)$ are nilpotent all formal power series terminate. Given an even element $\omega \in A_{p,q,s}$ with the vanishing free term one can construct a map $\alpha_\Lambda^\omega : \mathbf{R}_\Lambda^{p+q|s} \rightarrow \mathbf{R}^{0+1|0}$ using the same substitutions as those described above. It can be readily verified that the mappings α_Λ^ω for different Λ are consistent, i.e. the corresponding diagrams of the form (7) are commutative, and thus define a mapping of superspace $\mathbf{R}^{p+q|s}$ into the superspace $\mathbf{R}^{1|0}$, $\mathbf{R}^{0|1}$ or $\mathbf{R}^{0+1|0}$ if ω is even, odd or even and nilpotent respectively. In the same fashion, given a row L consisting of p' even, q' even and nilpotent and s' odd elements from $A_{p,q,s}$ one

can construct a map $\alpha_\Lambda^L : \mathbf{R}_\Lambda^{p \oplus q | s} \rightarrow \mathbf{R}_\Lambda^{p' \oplus q' | s'}$. The maps α_Λ^L for different Λ are mutually consistent and determine a map α^L of the corresponding superspaces. In fact maps of the form α^L exhaust all maps between superspaces $\mathbf{R}^{p \oplus q | s}$ and $\mathbf{R}^{p' \oplus q' | s'}$. More precisely, we have

Theorem 1. *The constructed correspondence of rows of p' even, q' even and nilpotent and s' odd elements from $A_{p,q,s}$ to mappings $\mathbf{R}^{p \oplus q | s} \rightarrow \mathbf{R}^{p' \oplus q' | s'}$ is a bijection.*

Proof. The injectivity follows directly from the construction of correspondence. Thus we only need to show that the map is onto. Assume the mappings $\alpha_\Lambda : \mathbf{R}_\Lambda^{p \oplus q | s} \rightarrow \mathbf{R}_\Lambda^{p' \oplus q' | s'}$ define a mapping of corresponding superspaces. Let $\Lambda_{k,r,N}$ be an AN algebra generated by the even generators a_1, \dots, a_k satisfying the relations $a_1^N = a_2^N = \dots = a_k^N = 0$ and odd generators η_1, \dots, η_r . We assume the only relations in $\Lambda_{k,s,N}$ are due to supercommutativity and those relations on even generators that are written above. Let Λ be any AN algebra and $x = (m_1 + n_1, \dots, m_p + n_p, l_1, \dots, l_q, \theta_1, \dots, \theta_s)$ be any point from $\mathbf{R}_\Lambda^{p+q | s}$ (here $m_i + n_i$ is the decomposition into numerical and nilpotent parts respectively). Denote by $\tau_{m,N}$ a point $(m_1 + a_1, \dots, m_p + a_p, a_{p+1}, \dots, a_{p+q}, \eta_1, \dots, \eta_s) \in \mathbf{R}_{\Lambda_{p+q,s,N}}^{p \oplus q | s}$ (the lower script m stands for the collection (m_1, \dots, m_p)). We claim that if N is sufficiently large there exists a morphism $\rho : \Lambda_{p+q,s,N} \rightarrow \Lambda$ such that $\tilde{\rho}(\tau_{m,N}) = x$. Indeed, let $n(\Lambda)$ be the maximal nilpotency degree of elements in Λ . Then, for $N \geq n(\Lambda)$ the map

$$\begin{aligned} a_1 &\mapsto n_1, \dots, a_p \mapsto n_p \\ a_{p+1} &\mapsto l_1, \dots, a_{p+q} \mapsto l_q \\ \eta_1 &\mapsto \theta_1, \dots, \eta_s \mapsto \theta_s \end{aligned}$$

can be extended up to a parity preserving homomorphism $\rho : \Lambda_{p+q,s,N} \rightarrow \Lambda$. One can easily see that for ρ constructed this way we have $\tilde{\rho}(\tau_{m,N}) = x$. By commutativity of diagram (7) corresponding to the homomorphism ρ we have $\tilde{\rho} \alpha_{\Lambda_{p+q,s,N}} = \alpha_\Lambda \tilde{\rho}$. Therefore $\alpha_\Lambda(x) = \alpha_\Lambda \tilde{\rho}(\tau_{m,N}) = \tilde{\rho} \alpha_{\Lambda_{p+q,s,N}}(\tau_{m,N})$ which means that the image of x under the map α_Λ is uniquely determined by the image of τ_m under the map $\alpha_{\Lambda_{p+q,s,N}}$. Since Λ is an arbitrary AN algebra we see that the mapping of superspaces α is determined by the images of points $\tau_{m,N} \in \mathbf{R}_{\Lambda_{p+q,s,N}}^{p \oplus q | s}$ for different N . Since the nilpotent parts of the coordinates of $\tau_{m,N}$ are the generators of $\Lambda_{p+q,s,N}$ the coordinates of the image of $\tau_{m,N}$ under $\alpha_{\Lambda_{p+q,s,N}}$ are clearly polynomials in them with coefficients depending on m_1, \dots, m_p and hence define elements from $A_{p,q,s}$. The last step is to show that the polynomials corresponding to different N are consistent in a sense that polynomials corresponding to algebras with smaller N are just initial pieces of those corresponding to the algebras with larger N . This is due to the fact that there exists a homomorphism $\rho : \Lambda_{p+q,s,M} \rightarrow \Lambda_{p+q,s,N}$ for $M \geq N$ such that

$\tilde{\rho}(\tau_{m,M}) = \tau_{m,N}$. The homomorphism ρ is defined by the following maps

$$\tilde{a}_i \mapsto a_i, \quad \tilde{\eta}_j \mapsto \eta_j$$

where \tilde{a}_i, a_i are even and $\tilde{\eta}_j, \eta_j$ are odd generators of $\Lambda_{p+q,s,M}$ and $\Lambda_{p+q,s,N}$ respectively. The same use of commutative diagram (7) for thus constructed ρ as above assures that the polynomials corresponding to $\alpha_{\Lambda_{p+q,s,N}}(\tau_m)$ are consistent and thus define in a unique way a row of p' even, q' even and nilpotent and s' odd elements from $A_{p,q,s}$. \square

Let us turn now to some additional requirements that we impose on admissible mappings of supermanifolds. First note that if Λ is an AN algebra such that $\dim \Lambda_0 = k, \dim \Lambda_1 = l$ then we have an isomorphism $\mathbf{R}_\Lambda^{p \oplus q|s} \cong \mathbf{R}^L$, $L = pk + q(k-1) + sl$. Since the tangent space TR^L is isomorphic to \mathbf{R}^L itself, for every Λ point $x \in \mathbf{R}_\Lambda^{p \oplus q|s}$ we have a Λ_0 module structure induced on T_x in a natural way. We say that a smooth mapping $\alpha_\Lambda : \mathbf{R}_\Lambda^{p \oplus q|s} \rightarrow \mathbf{R}_\Lambda^{p' \oplus q'|s'}$ is Λ_0 -smooth if for every $x \in \mathbf{R}_\Lambda^{p \oplus q|s}$ the tangent map $(\alpha_\Lambda)_* : T_x \rightarrow T_{\alpha_\Lambda(x)}$ is a homomorphism of Λ_0 modules. The mapping $\alpha : \mathbf{R}^{p \oplus q|s} \rightarrow \mathbf{R}^{p' \oplus q'|s'}$ is said to be smooth if for all AN algebras Λ the maps α_Λ are Λ_0 -smooth. The description of smooth mappings between the spaces $\mathbf{R}^{p \oplus q|s}$ is given by theorem 2 which we will formulate in a moment. But first we introduce the algebra $B_{p,q,s}$ as a tensor product of $C^\infty(\mathbf{R}^p)$, an algebra of formal series with respect to q variables n_1, \dots, n_q and a Grassmann algebra with s generators ξ_1, \dots, ξ_s :

$$B_{p,q,s} = C^\infty(\mathbf{R}^p) \otimes \mathbf{R}[[n_1, \dots, n_q]] \otimes \Lambda(\xi_1, \dots, \xi_s)$$

More explicitly, elements of $B_{p,q,s}$ are of the form (8) where coefficients $f^{\alpha_1, \dots, \alpha_i}$ are series of the form

$$f^{\alpha_1, \dots, \alpha_i} = \sum_{(\beta_1, \dots, \beta_q)} G_{\beta_1, \dots, \beta_q}^{\alpha_1, \dots, \alpha_i}(x_1, \dots, x_p) l_1^{\beta_1} \dots l_p^{\beta_q} \quad (10)$$

and $G_{\beta_1, \dots, \beta_q}^{\alpha_1, \dots, \alpha_i}(x_1, \dots, x_p)$ are smooth functions of variables x_1, \dots, x_p . Again the Z_2 grading in the exterior algebra $\Lambda(\xi_1, \dots, \xi_s)$ induces a Z_2 grading on $B_{p,q,s}$. One can check that given a Λ -point from $\mathbf{R}^{p \oplus q|s}$ and an even (odd) element from $B_{p,q,s}$ by substituting first p coordinates instead of variables x_1, \dots, x_p , next q coordinates instead of variables l_1, \dots, l_q and the last s variables instead of ξ_1, \dots, ξ_s one gets an even (odd) element from Λ . These substitutions make sense because one can use the Taylor expansion of functions G in the nilpotent parts of the first p variables and the formal series in the next q variables terminate due to nilpotency. In parallel with the case of algebra $A_{p,q,s}$ one can check that a row of $p' + q' + s'$ elements from $B_{p,q,s}$ with appropriate parities and nilpotency properties defines a mapping $\mathbf{R}^{p \oplus q|s} \rightarrow \mathbf{R}^{p' \oplus q'|s'}$. Moreover, this map is smooth in the sense explained above. Note that such a row also defines a parity preserving homomorphism $B_{p',q',s'} \rightarrow B_{p,q,s}$ (the generators are sent to the corresponding elements of the row). In fact the following theorem is valid:

Theorem 2. The smooth mappings $\mathbf{R}^{p \oplus q|s} \rightarrow \mathbf{R}^{p' \oplus q'|s'}$ are in a bijective correspondence with the parity preserving homomorphisms $B_{p',q',s'} \rightarrow B_{p,q,s}$.

Proof. We have shown how to construct a smooth mapping $\mathbf{R}^{p \oplus q|s} \rightarrow \mathbf{R}^{p' \oplus q'|s'}$ out of a row of p' even, q' even and nilpotent and s' odd elements from $B_{p,q,s}$. Conversely, given a mapping $\alpha : \mathbf{R}^{p \oplus q|s} \rightarrow \mathbf{R}^{p' \oplus q'|s'}$ by theorem 1 we can assign to it a row of elements from $A_{p,q,s}$. Let $\omega_i = \omega_i(m, n, l, \xi)$ (see (8), (9)) be the i -th coordinate of this row. From the assumption of Λ_0 -smoothness it follows directly that the coefficients $G_{\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_q}^{\alpha_1, \dots, \alpha_j}(m_1, \dots, m_p)$ corresponding to elements ω_i are smooth functions.

Let $\lambda = m(\lambda) + n(\lambda)$ be an element of Λ_0 . Set $m(\lambda)\epsilon$ and $n(\lambda)\epsilon$ to be an increment of numerical and nilpotent parts of the k -th coordinate of the space $\mathbf{R}_\Lambda^{p \oplus q|s}$ respectively ($k \leq p$, ϵ is a small real number). All other coordinates except the k -th one are fixed. Then, by Λ_0 -smoothness

$$\omega_i(m_k + m(\lambda)\epsilon, n_k + n(\lambda)\epsilon) - \omega_i(m_k, n_k) = \lambda(\omega(m_k + \epsilon, n_k) - \omega(m_k, n_k)) + o(\epsilon)$$

On the other hand the LHS of the last equation can be written as

$$m(\lambda)\epsilon \frac{\partial \omega_i(m_k, n_k)}{\partial m_k} + n(\lambda)\epsilon \frac{\partial \omega_i(m_k, n_k)}{\partial n_k} + o(\epsilon)$$

from which we conclude that $\frac{\partial \omega_i(m_k, n_k)}{\partial n_k} = \frac{\partial \omega_i(m_k, n_k)}{\partial m_k}$. The last equation being written in terms of series $\omega_i = b_0 + b_1 n_k + b_2 n_k^2 + \dots$ where b_i depend on m, l, ξ and all n 's except n_k reads as

$$b_1 + 2b_2 n_k + 3b_3 n_k^2 + \dots = \frac{\partial b_0}{\partial m_k} + \frac{\partial b_1}{\partial m_k} n_k + \frac{\partial b_2}{\partial m_k} n_k^2 + \dots$$

Therefore,

$$\omega_i = b_0 + \frac{1}{1!} \frac{\partial b_0}{\partial m_k} n_k + \frac{1}{2!} \frac{\partial^2 b_0}{\partial m_k^2} n_k^2 + \dots$$

Hence $\omega_i = b_0(m_k + n_k)$ where for shortness all other arguments are skipped. Repeating this argument successively for all $k = 1, \dots, p$ we get

$$\omega_j = \sum_{i=0}^s \sum_{1 \leq \alpha_1 < \dots < \alpha_i \leq s} f_j^{\alpha_1, \dots, \alpha_i} \xi_{\alpha_1} \dots \xi_{\alpha_i}$$

where

$$f_j^{\alpha_1, \dots, \alpha_i} = \sum_{(\gamma_1, \dots, \gamma_q)} G_{j; 0, \dots, 0, \gamma_1, \dots, \gamma_q}^{\alpha_1, \dots, \alpha_i}(m_1 + n_1, \dots, m_p + n_p) l_1^{\gamma_1} \dots l_q^{\gamma_q}$$

Compare with (8), (10). The last two expressions mean precisely that ω is given by a row of $p' + q' + s'$ elements (with corresponding parity and nilpotency properties) from the algebra $B_{p,q,s}$, i.e. represents a parity preserving homomorphism $B_{p',q',s'} \rightarrow B_{p,q,s}$. \square

The theorems 1 and 2 are modifications of statements proved in [4].

The theorem 2 has a straightforward generalization to the case of mappings between superdomains. Let $B_{p,q,s}(U) = \mathbf{R}[[n_1, \dots, n_q]] \otimes C^\infty(U) \otimes \Lambda \cdot (\xi_1, \dots, \xi_s)$. Then, we claim that smooth mappings of superdomains $U^{p+q,s} \rightarrow V^{p'+q',s'}$ are in a bijective correspondence with parity preserving homomorphisms $B_{p',q',s'}(V) \rightarrow B_{p,q,s}(U)$ (one can easily modify the proof taking into account the restriction on the range of first p (respectively p') coordinates). Now let us give a definition of supermanifold.

Definition. A $(p \oplus q|s)$ -dimensional supermanifold S is a superspace which is glued together from $(p \oplus q|s)$ -dimensional superdomains by means of smooth mappings.

In more detail, we assume that there exists a covering \mathcal{U} of the body $S_{\mathbf{R}}$ of superspace S with the property that a restriction of S to any $U \in \mathcal{U}$ which we denote as S^U is isomorphic to the superdomain $U_\alpha^{p \oplus q|s}$. Let U and V be any two intersecting sets from the covering \mathcal{U} and let $\phi_U : S^U \rightarrow U_\alpha^{p \oplus q|s}$, $\phi_V : S^V \rightarrow V_\alpha^{p \oplus q|s}$ be the corresponding isomorphisms. Denote by $\tilde{\phi}_U, \tilde{\phi}_V$ the isomorphisms $S^{U \cap V} \rightarrow (U \cap V)_\alpha^{p \oplus q|s}$ induced by ϕ_U and ϕ_V respectively. Then we require that the mappings $(\tilde{\phi}_U)^{-1} \tilde{\phi}_V$ are smooth.

Using theorem 2 one can give a definition of a $(p \oplus q|s)$ -dimensional supermanifold along the lines of a conventional Berezin-Leites approach to supermanifolds via ringed spaces. In those terms, a $(p \oplus q|s)$ -dimensional supermanifold is a manifold \mathcal{M}_0 with a sheaf \mathcal{O} of supercommutative rings on it with the property that locally over a neighborhood U the sheaf is isomorphic to the algebra $B_{p,q,s}(U) = C^\infty(U) \otimes \mathbf{R}[[n_1, \dots, n_q]] \otimes \Lambda \cdot (\xi_1, \dots, \xi_s)$.

On the other hand, given a supermanifold \mathcal{M} one can construct a sheaf of Z_2 -graded supercommutative algebras over the body \mathcal{M}_0 as follows. For any open subset $U \subset \mathcal{M}_0$ consider the space of mappings from the restriction of \mathcal{M} to U into $\mathbf{R}^{1|1}$. Denote this space by $\mathcal{F}(U)$. Note that one can identify $\mathbf{R}_\Lambda^{1|1}$ with Λ . This induces a structure of Z_2 -graded supercommutative algebra on $\mathcal{F}(U)$. It is easy to see that the collection of $\mathcal{F}(U)$ is a (pre)sheaf endowing \mathcal{M} with a structure of ringed space. The fact that locally our supermanifold is isomorphic to a superdomain determines the standard local structure of the sheaf at hand.

We define a tangent bundle $T\mathcal{M}$ of a supermanifold \mathcal{M} as a functor determined by Λ -points $(T\mathcal{M})_\Lambda = T(\mathcal{M}_\Lambda)$ and mappings $\tilde{\alpha}_*$ corresponding to morphisms $\alpha : \Lambda \rightarrow \Lambda'$. One can easily check that $T\mathcal{M}$ is a supermanifold itself and has a canonical projection $T\mathcal{M} \rightarrow \mathcal{M}$.

It is possible to consider superspaces modelled on a fixed infinite-dimensional topological linear superspace V . Such a superspace V can be constructed as in Example 2 out of any infinite-dimensional topological Z_2 -graded linear space. An infinite-dimensional superdomain is a restriction of V to any open subset of its body. In the infinite-dimensional case one can also define objects analogous to

$(k \oplus l|q)$ -dimensional linear superspaces. From now on let us fix V . One can give different definitions of differentiable (smooth) mappings between two topological linear spaces. For our purposes the following definition will be sufficient. We will call a mapping $F : V_1 \rightarrow V_2$ between two infinite-dimensional topological linear superspaces differentiable if there exists a mapping $D : V_1 \times V_1 \rightarrow V_2$ such that D is linear in the second variable and for each AN algebra Λ $F_\Lambda(x+h) = F_\Lambda(x) + D_\Lambda(x, h) + o_x(h)$ for any $x, h \in (V_1)_\Lambda$. Here $o_x(h)$ vanishes to an order higher than h , i.e. for a real number $t \rightarrow 0$, $t^{-1}o_x(th) \rightarrow 0$. (this is what is called differentiability in the sense of Gâteaux, see for example [9]). Once we defined a differentiable mapping between superdomains in V we can consider infinite-dimensional differentiable supermanifolds modelled on V as superspaces pasted from superdomains in V by means of differentiable mappings.

Lie supergroups. We have already given above the definitions of supergroup and supermanifold. Following the same category-theoretic point of view we define a $(p \oplus q|s)$ -dimensional Lie supergroup as a $(p \oplus q|s)$ -dimensional supermanifold which is also a supergroup and for each AN algebra Λ the set of Λ -points is a Lie group with respect to the given smooth manifold and group structures. To any $(p \oplus q|s)$ -dimensional Lie supergroup one assigns naturally its Lie superalgebra being a $(p \oplus q|s)$ -dimensional vector space. All this can be generalized easily to include infinite-dimensional Lie supergroups modelled on a linear superspace V .

Given any Lie superalgebra \mathcal{G} (possibly infinite-dimensional) one can consider its restriction to a point 0 of the body and obtain a Lie superalgebra \mathcal{G}_{nilp} which has only nilpotent coordinates. Then, for each AN algebra Λ one can consider a set of formal power series $G_\Lambda = \{exp(a) | a \in (\mathcal{G}_{nilp})_\Lambda\}$. The set G_Λ is endowed with a Lie supergroup structure via the Campbell-Hausdorff formula which is represented by a finite sum because of nilpotency. It is easy to check that the sets G_Λ constitute a Lie supergroup G_{nilp} modelled on a linear superspace $V = \mathcal{G}_{nilp}$ and having \mathcal{G}_{nilp} as its Lie superalgebra. If there exists a Lie supergroup G having \mathcal{G} as its Lie superalgebra, then G_{nilp} can alternatively be constructed as the restriction of superspace G to the subset of its body $G^{(0)}$ consisting of one point - the unit element of Lie group $G^{(0)}$. These two constructions are canonically isomorphic (the isomorphism is set up via exponential mappings in G_Λ 's). In a more general situation it might happen that \mathcal{G} contains an invariant Lie subalgebra $\mathcal{G}_{int} \subset \mathcal{G}$ such that there exists a Lie supergroup G_{int} with $\mathcal{G}_{int} = Lie(G_{int})$ (the subscript "int" comes from the word "integrable"). Then, the construction above can be improved as follows.

First note that the supergroup G_{int} acts on G_{nilp} by the adjoint action on \mathcal{G}_{nilp} . Then we can consider a semidirect product of Lie supergroups $G_{nilp} \times_{Ad} G_{int}$, i.e. the direct product supermanifold $G_{nilp} \times G_{int}$ with a group operation defined by the following formula

$$(e^x, a)(e^y, b) = (e^x(ae^y a^{-1}), ab)$$

As it was explained G_{int} contains $(G_{int})_{nilp}$ which is canonically isomorphic

to a Lie subgroup $(G_{int})_{nilp} \subset G_{nilp}$. It is straightforward to check that the set of pairs $H = \{(e^x, e^{-x}) \in G_{nilp} \times G_{int} | e^x \in (G_{int})_{nilp}\}$ is an invariant Lie subgroup of $G_{nilp} \times_{Ad} G_{int}$. Taking the quotient with respect to H we get a Lie supergroup whose Lie superalgebra is isomorphic to the one obtained as a restriction of Lie superalgebra \mathcal{G} to a subset of its body $\mathcal{G}_{int}^{(0)} \subset \mathcal{G}^{(0)}$.

Superspaces of maps. In this section we introduce a superspace of maps and give some explicit constructions of this object. Let \mathcal{E}, \mathcal{F} be superspaces. For any superspace \mathcal{S} a mapping $\sigma : \mathcal{E} \times \mathcal{S} \rightarrow \mathcal{F}$ is called a family of mappings from the superspace \mathcal{E} into the superspace \mathcal{F} with base \mathcal{S} . A family of mappings $\nu : \mathcal{E} \times \mathcal{F}^{\mathcal{E}} \rightarrow \mathcal{F}$ is called a universal family of mappings $\mathcal{E} \rightarrow \mathcal{F}$ if it has the following property. For any family $\mathcal{E} \times \mathcal{S} \rightarrow \mathcal{F}$ there exists a unique mapping $\rho : \mathcal{S} \rightarrow \mathcal{F}^{\mathcal{E}}$ such that $\nu \circ (id_{\mathcal{E}} \times \rho) = \sigma$. In the case when \mathcal{E}, \mathcal{F} are supermanifolds we assume that all mappings are smooth and the base \mathcal{S} of the family is a supermanifold though the base of the universal family need not be a supermanifold. For any particular pair of superspaces \mathcal{E}, \mathcal{F} there arises naturally a question of whether such a universal family of mappings exists. Provided a universal family exists and is unique up to an isomorphism, the base of the universal family is said to be the superspace of mappings (between corresponding superspaces). Below we will construct superspaces of maps between superdomains and describe a superspace of maps for the case of supermanifolds.

Example. In this example we give a construction of the superspace of smooth maps $\mathbf{R}^{p|q} \rightarrow \mathbf{R}^{p'|q'}$. Due to theorem 1 from section 3 the smooth mappings $\mathbf{R}^{p|q} \rightarrow \mathbf{R}^{p'|q'}$ are in a bijective correspondence with parity preserving homomorphisms $B_{p',q'} \rightarrow B_{p,q}$ ($B_{p,q} \equiv B_{p,q,0}$). These homomorphisms form a vector space $V_{p',q'}^{p,q}$ whose elements are rows of p' even and q' odd elements from $B_{p,q}$. Likewise one can consider a vector space $\tilde{V}_{p',q'}^{p,q}$ of parity reversing homomorphisms. Its elements are rows of p' odd and q' even elements from $B_{p,q}$. Define a vector superspace $\mathbf{V}_{p',q'}^{p,q}$ as a vector superspace associated with a \mathbb{Z}_2 -graded vector space $V_{p',q'}^{p,q} \oplus \tilde{V}_{p',q'}^{p,q}$. We assert that $\mathbf{V}_{p',q'}^{p,q}$ can be considered as a superspace of smooth mappings $\mathbf{R}^{p|q} \rightarrow \mathbf{R}^{p'|q'}$. To an element $b \in \mathbf{V}_{p',q'}^{p,q}(\Lambda)$ we assign a smooth mapping $\mathbf{R}_{\Lambda}^{p|q} \rightarrow \mathbf{R}_{\Lambda}^{p'|q'}$ in a natural way. One can check that this gives a family of smooth mappings $\nu_{p',q'}^{p,q} : \mathbf{R}^{p|q} \times \mathbf{V}_{p',q'}^{p,q} \rightarrow \mathbf{R}^{p'|q'}$. Now let $\sigma : \mathbf{R}^{p|q} \times S \rightarrow \mathbf{R}^{p'|q'}$ be a family of smooth mappings. The most important case is $S = \mathbf{R}^{m|n}$. Then, $\mathbf{R}^{p|q} \times S \cong \mathbf{R}^{p+m|q+n}$ and by Theorem 2, σ corresponds to an element from $V_{p',q'}^{p+m,q+n}$. This element can be considered as a row of $p' + q'$ elements $\sigma^k, k = 1, 2, \dots, p' + q'$ from $B_{p+m,q+n}$ with the corresponding parity. Explicitly

$$\sigma^k = \sum_{i=0}^q \sum_{j=0}^n \sum_{\alpha, \beta} f_{\alpha_1, \dots, \alpha_i, \beta_1, \dots, \beta_j}^k(x^1, \dots, x^p, y^1, \dots, y^m) \xi^{\alpha_1} \dots \xi^{\alpha_i} \eta^{\beta_1} \dots \eta^{\beta_j}$$

where we assume that variables x^i, ξ^j (y^i, ξ^i) are assigned to the superspace $\mathbf{R}^{p|q}$ (respectively $\mathbf{R}^{m|n}$) and the summation runs over the numbers i and j such that $i + j$ is of the appropriate parity. The collection of elements from $B_{p,q}$

$$g^k(x^1, \dots, x^p)_{\alpha_1, \dots, \alpha_i} = \sum_{j=0}^n \sum_{1 \leq \beta_1 < \dots < \beta_j \leq n} f_{\alpha_1, \dots, \alpha_i, \beta_1, \dots, \beta_j}^k(x^1, \dots, x^p, y^1, \dots, y^m) \eta^{\beta_1} \dots \eta^{\beta_j}$$

labeled by the sequence of multiindices $1 \leq \alpha_1 < \dots < \alpha_i \leq q$, integer k , and the variables x^1, \dots, x^p defines a mapping $\rho : \mathbf{R}^{m|n} \rightarrow \mathbf{V}_{p',q'}^{p,q}$. One should simply utilize the standard substitutions of Λ -point coordinates into $g^k(x^1, \dots, x^p)_{\alpha_1, \dots, \alpha_i}$ and then form a combination

$$\sum_{i=0}^q \sum_{1 \leq \alpha_1 < \dots < \alpha_i \leq q} \xi^{\alpha_1} \dots \xi^{\alpha_i} g^k(x^1, \dots, x^p)_{\alpha_1, \dots, \alpha_i} \in \mathbf{V}_{p',q'}^{p,q}(\Lambda)$$

A direct check shows that $\nu_{p',q'}^{p,q} \circ \rho = \sigma$. Using the fact that locally every supermanifold is equivalent to a superdomain one can prove that $\mathbf{V}_{p',q'}^{p,q}$ is indeed a superspace of maps. Note that the same considerations as above go through for the case of mappings $U^{p|q} \rightarrow \mathbf{R}^{p'|q'}$ of a superdomain to a linear superspace. One simply needs to consider vector spaces of parity preserving and parity reversing homomorphisms $B_{p',q'} \rightarrow B_{p,q}(U)$ and repeat the construction above. The superspace of maps between superdomains $U^{p|q} \rightarrow V^{p'|q'}$ is the restriction of superspace of maps $U^{p|q} \rightarrow \mathbf{R}^{p'|q'}$ to the subset of its body consisting of maps carrying the body of the superdomain U to the subset $V \subset \mathbf{R}^{p'|q'}$. The generalization of these results to the case of $(p \oplus q|s)$ -dimensional superdomains is straightforward. One should replace everywhere algebras of the type $B_{p,q}$ by the algebras of the type $B_{p,q,s}$.

Now let us discuss briefly a situation in the case of mappings between two supermanifolds. Let \mathcal{M} and \mathcal{N} be supermanifolds. Then, the restriction of superspace of maps $\mathcal{N}^{\mathcal{M}}$ to each connected component of its body can be endowed with a structure of infinite-dimensional supermanifold. For a given map $\phi : \mathcal{M} \rightarrow \mathcal{N}$ consider the pull-back of the tangent bundle $T\mathcal{N}$ induced by ϕ . Denote this bundle as $\phi^*T\mathcal{N}$. The superspace of sections of $\phi^*T\mathcal{N}$ is a vector superspace which can be regarded as a tangent space to $\mathcal{N}^{\mathcal{M}}$ at the point ϕ in a natural way. Putting some (super)Riemannian metric on \mathcal{N} one can extend the infinitesimal variation of ϕ corresponding to a given section of $\phi^*T\mathcal{N}$ to some finite variation. This construction determines a neighborhood of the mapping ϕ being isomorphic to some open set (the restriction to some open subset of the body) of the vector superspace of sections $\Gamma(\phi^*T\mathcal{N})$.

G-structures on supermanifolds. Our next goal is to introduce the notion of G-structure on a supermanifold. Let \mathcal{M} be a supermanifold of dimension

$(m \oplus n|q)$ and let \mathcal{U} be a covering of the body \mathcal{M}_R such that the restriction of \mathcal{M} to any $U \in \mathcal{U}$ is isomorphic to a superdomain $U^{m \oplus n|q}$. For any $U, V \in \mathcal{U}$ denote by $\phi_{V,U}$ the corresponding gluing transformation (from the chart U to V). We say that $L(\mathcal{M})$ is (the total space of) the frame bundle over \mathcal{M} if $L(\mathcal{M})$ is a supermanifold glued together from the superdomains $U^{m \oplus n|q} \times GL(m \oplus n|q)$ by means of transformations $\phi_{V,U} \times J$ where J is the left multiplication by the jacobian matrix corresponding to $\phi_{U,V}$. Explicitly, if $X = (x^1, \dots, x^m, y^1, \dots, y^n, \theta^1, \dots, \theta^q)$ are coordinates on the superdomain $U^{m \oplus n|q}$ then the gluing transformation from the chart $U^{m \oplus n|q} \times GL(m \oplus n|q)$ to the chart $V^{m \oplus n|q} \times GL(m \oplus n|q)$ acts on the element $(a, g) \in U^{m \oplus n|q} \times GL(m \oplus n|q)$ as follows: $(a, g) \mapsto (\phi_{V,U}(a), \frac{\partial \phi_{V,U}}{\partial X}(a)g)$. The supermanifold $L(\mathcal{M})$ constructed this way is equipped with a natural structure of principal $GL(m \oplus n|q)$ bundle over \mathcal{M} . One can show that $L(\mathcal{M})$ does not depend on the choice of covering \mathcal{U} . Let G be a Lie (super)subgroup of $GL(m \oplus n|q)$ then, any smooth mapping $\mathcal{M} \rightarrow G$ acts naturally from the right on $L(\mathcal{M})$. Furthermore, the superspace of maps $G^{\mathcal{M}}$ can be furnished with a natural supergroup structure (in a “point-wise” manner). This supergroup acts on $L(\mathcal{M})$. Taking the quotient space with respect to this action one obtains a superspace denoted by $L(\mathcal{M})/G^{\mathcal{M}}$. Since the elements of $G^{\mathcal{M}}$ act fiberwise, the superspace $L(\mathcal{M})/G^{\mathcal{M}}$ is equipped with a structure of fiber bundle over \mathcal{M} . For any Lie (super)subgroup $G \subset GL(m \oplus n|q)$ we say that there is a G -structure defined on a supermanifold \mathcal{M} if a section of the bundle $L(\mathcal{M})/G^{\mathcal{M}} \rightarrow \mathcal{M}$ is specified. In general, given a bundle $p: \mathcal{E} \rightarrow \mathcal{B}$ where \mathcal{E}, \mathcal{B} one can consider the corresponding superspace of sections denoted by $\Gamma(\mathcal{E})$ as a subspace in $\mathcal{E}^{\mathcal{B}}$. Thus we can consider the superspace of sections of the bundle $L(\mathcal{M})/G^{\mathcal{M}} \rightarrow \mathcal{M}$. We call this superspace the superspace of G -structures over \mathcal{M} . The supergroup of diffeomorphisms of \mathcal{M} acts naturally on the superspace of G -structures over \mathcal{M} . The corresponding quotient space is called the moduli space of G -structures. Note that this moduli space is a superspace by definition but in general it is not a supermanifold. In particular situations the most interesting case is that of locally standard G -structures. For a G -structure to be locally standard we mean the following. Fix a G -structure B_{st} on $\mathbf{R}^{m+n|q}$ that will be called standard. Note that a G -structure on supermanifold \mathcal{M} induces a G -structure on any superspace obtained by restriction to a subset of the body $U \subset \mathcal{M}_R$. Assume \mathcal{U} is a standard atlas for the manifold \mathcal{M} , i.e. \mathcal{M} restricted to any element from \mathcal{U} is equivalent to a superdomain. Then, a G structure B on \mathcal{M} is called locally standard if the restriction of B to any neighborhood $U \in \mathcal{U}$ is isomorphic to the restriction of B_{st} to U .

A rigorous definition of (p, q) -superconformal manifold can also be given in the language of G -structures. This would provide a rigorous definition for the moduli space of $(k \oplus l|q)$ -dimensional supermanifolds as was outlined above for the general case of arbitrary G -structure.

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